NOTE ON THE EXISTENCE OF A MINIMUM OF $\int_{x_0y_0}^{x_1y_1} Pdx + Qdy$.

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In attempting to apply the principles of calculus of variations to the integral

$$J = \int_{x_0y_0}^{x_1y_1} \big(P(xy) + y'Q(xy) \big) dx,$$

Euler's differential equation degenerates into the finite equation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1}$$

This case is usually dismissed with the remark that if (1) is an identity the integral is independent of the path, while if it is not an identity the given points (x_0y_0), (x_1y_1) will not lie on the curve defined by it. It is the purpose of this note to find the conditions for a minimum (maximum) in the case that the end points actually lie on this curve. Call the curve defined by (1), C, and assume that all the second partial derivatives of P and Q exist and are continuous in a neighborhood of C, to use Bolza's notation, that P and Q are of class C'' in this neighborhood.

If we form the second variation of J taken along C and apply to it Legendre's transformation, we obtain it in the form

$$\delta^2 J = \frac{\epsilon^2}{2} \int_{x_0}^{x_1} \eta^2 \left\{ \frac{\partial (P_y - Q_x)}{\partial y} \right\} dx;$$

so that evidently it is necessary for a minimum that

$$\frac{\partial (P_y - Q_x)}{\partial y} \geqq 0$$

along C.

If the stronger condition is satisfied, C has no tangent parallel to the Y-axis in the interval under consideration, and the function $P_y - Q_x$, which vanishes along the curve C, is negative below and positive above it. I shall assume that this latter is the case. Then C actually makes J a minimum.

To prove this we have merely to draw any other curve $\bar{C}$ connecting 0 and 1 and lying in the neighborhood of C. For convenience assume it lies entirely above C. The extension to other cases is immediate. Consider the line integral $\int (P + y'Q)dx$ taken along the contour formed by the

curves C and $\bar{C}$ in positive sense. By Green's theorem, which applies under the above assumptions,

$$\int_{C,\bar{C}} (P + y'Q)dx = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

where the latter integral is to be taken over the area bounded by C and $\bar{C}$. But this latter integral is negative since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is negative above C according to our assumptions. Then $J_C - J_{\bar{C}} < 0$ or $J_C < J_{\bar{C}}$, which we wished to prove.

For example,* the work done by a force whose components are $F_x = ky^2$, $F_y = kxy$, is

$$W = k \int_{x_0 y_0}^{x_1 y_1} \{y^2 + xyy'\} dx.$$

Here

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = y, \qquad \frac{\partial}{\partial y}(P_y - Q_x) = 1 > 0.$$

Hence the above force will do the least work on a particle describing a path from $(a, 0)$ to $(b, 0)$, $b > a$, if the particle moves along the X-axis.

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* Smith and Longley, Theoretical Mechanics, p. 112, Ex. 4.